On leveled beta groups

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Abstract

In the paper the algebraic structure with infinitely many identity elements are studied. Differently from group of units in associative ring, identity elements in the beta group are defined axiomatically. The first attempt to construct such algebraic structure can be found in (Javtokas, 2006), but it was not successful due to the lack of examples. The present paper improves a concept of the beta group by defining leveled beta groups, in such a way, that it becomes possible to construct examples of such algebraic structures.

Keywords: identities, groups, groups of units, ordinal numbers.
MSC (2000): 20A05, 20N99, 03E10

1 Introduction

In group theory axiom system is chosen so, that we can easily prove uniqueness of identity or inverse element. But naturally emerges question: "Is it possible to define consistent mathematical structure with infinitely many non trivial identities? And is that new structure maintains group properties?" The beginning of investigation of such objects were begun in the book (Javtokas, 2006). The book suggested axiom system of the beta group, which can be viewed as some generalization of a group concept. We can identify some analogy between the beta group and a group of units in associative ring, where some recent developments (see Bovdi, 2005; Coleman and Easdown, 2000; Dekimpe, 2003; Giambruno, Jespers and Valenti, 1994; Giambruno, 1996; Hill, 1994; Jespers and Polcino, 1996; Kawai, 2005; Li and Parmenter, 2005; Pita, del Río and Ruiz, 2005; Szechtman, 2004;

Wilcox, 2004) covers broad range of topics, but our concept of many "units" is lying in the core of chosen axiomatic system, it is not definable in secondary steps. The most surprising fact about beta groups was that it maintains some of classical groups properties.

Let B be a set and $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ its subsets which in (Javtokas, 2006) was called a subset of identity elements, inverse elements and the rest elements respectively, and $B = B^{(1)} \cup B^{(2)} \cup B^{(3)}$.

Definition 1.1. *The beta group* $(B, *)$ *is a set* B *together with a binary operation* $*$ *satisfying the following axioms.*

- **BG 1**. *B is closed under the operation $*$, that is, $a * b \in B$ for all $a, b \in B$.*
- **BG 2**. *There are identity elements $e_i \in B^{(1)}$, such that $a * e_i = a$ for all $i \in \mathbb{N}$ and for all $a \in B^{(3)}$.*
- **BG 3**. *For each $e_i \in B^{(1)}$, there are $e_j \in B^{(1)}$, such that $e_i * e_j = e_j$ and $e_i \neq e_j$ $(i \neq j)$ for all $i, j \in \mathbb{N}$.*
- **BG 4**. *Each element $a \in B^{(3)}$ has inverse elements $\hat{a}_i \in B^{(2)}$ such that $a * \hat{a}_i = e_i$ for all $i \in \mathbb{N}$.*
- **BG 5**. *The operation $*$ is associative, that is $(a * b) * c = a * (b * c)$ for all $a, b, c \in B$, except if one of elements belongs to $B^{(1)}$ and other to $B^{(2)}$ with different indexes i.e. $a*(e_i*\hat{c}_j) \neq (a*e_i)*\hat{c}_j$.*

There are two principal keystones of the beta group. Firstly by **BG 2** we have infinitely many non trivial identity elements. Secondly, by **BG 4**, every element of $B^{(3)}$ has infinitely many non trivial inverses, i.e. $a*\hat{a}_i = e_i$, $b * \hat{b}_i = e_i$, $c * \hat{c}_i = e_i, \ldots$, for all $i \in \mathbb{N}$.

The main problem, which has emerged from the first steps of the theory of beta groups is finding an example, which would materialize a theoretical concept of beta groups. For this purpose we introduce a concept of leveled beta groups, and give example which actualize them.

2 Leveled beta groups

Before defining leveled beta groups let us introduce the following notations. Let us by $a^{(n}_{(b}$ denote the element a which belongs to the n-th level, that is $a^n \in B^{(n)}$, and depends from the element b. We will write $a^{(n}_{(b,c}$ if a depends from b and c. In general $a^{(n} \neq a^{(n}_{(b} \neq a^{(n}_{(b,c}$. By $\mathbb{I}$ we denote the index set.

Definition 2.1. *The first level beta group* $(B, *)$ *is a set* $B = B^{(1)}$ *together with a binary operation* $*$ *satisfying the following axioms.*

1° *B is closed under the operation $*$, that is, $a * b \in B$ for all $a, b \in B$.*

2° *For each $e_i \in B^{(1)}$, there are $e_j \in B^{(1)}$, such that $e_i * e_j = e_k$ and $e_i \neq e_j \neq e_k$ $(i \neq j)$ for every $i, j, k \in \mathbb{I}$.*

3° *The operation $*$ is associative, that is $(e_i * e_j) * e_k = e_i * (e_j * e_k)$ for all $e_i \in B^{(1)}$ and every $i, j, k \in \mathbb{I}$.*

An example of the first level beta group can be $(\mathbb{N}, +)$. From Definition 2.1 straight away follows that the first level beta group is a semigroup.

To shorten notations, let us denote $a^{(m}_{(i} := a^{(m}_{(e_i}$.

Definition 2.2. *The second level beta group $(B, *)$ is a set $B = B^{(1)} \cup B^{(2)}$ together with a binary operation $*$ satisfying the following axioms.*

1°-2° *Are the same as in Definition 2.1.*

3° *There are identity elements $e_i \in B^{(1)}$, such that $e_i * a^{(2} = a^{(2}$ and $a^{(2} * e_i = a^{(2}_{(i}$ for all $i \in \mathbb{I}$ and for all $a^{(2}, a^{(2}_{(i} \in B^{(2)}$.*

4° *The operation $*$ is associative, that is $(a * b) * c = a * (b * c)$ for all $a, b, c \in B$.*

To construct an example for the second level beta group we need some definitions from classical set theory. By lowercase Greek letters we shall denote ordinal numbers. All the basic properties of ordinals like associativity, non-commutativity, ..., can be found in (Jech, 2002).

If $\alpha = \beta + 1$, then α is a successor ordinal. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$; α is called a limit ordinal. We denote the least nonzero limit ordinal ω.

As usual, by Ord we denote the class of all ordinal numbers. Let $\mathcal{O}$ be a subset of Ord, which is not less than $\mathbb{N}$, for example, $\mathcal{O} = \{\omega, \omega+1, \ldots, \omega+\omega, \omega+\omega+1, \ldots\}$. Then, as an example of the second level β-group could be $(\mathbb{N} \cup \mathcal{O}, +)$. As we see from Definition 2.1 the first level beta groups consist of one kind of elements. In the second level beta groups emerge new set of elements $B^{(2)}$ for which all the elements from the first level are identities (from the left), and inverse elements $a^{(2}_{(i}$, if the algebraic operation is performed from the right.

Definition 2.3. *The third level beta group $(B, *)$ is a set $B = B^{(1)} \cup B^{(2)} \cup B^{(3)}$ together with a binary operation $*$ satisfying the following axioms.*

1°-4° *Are the same as in Definition 2.2.*

5° *Each element* $c^{(3} \in B^{(3)}$ *has identity elements* $a^{(2} \in B^{(2)}$*, such that* $a^{(2} * c^{(3} = c^{(3}$ *and inverse elements* $c^{(3}_{(a} \in B^{(3)}$*, such that* $c^{(3} * a^{(2} = c^{(3}_{(a}$*.*

Now we have three kinds of sets. It is easy to see that it's impossible to get from one set to another by performing any amount of algebraic operations between elements belonging to any one of $B^{(i)}$ sets, i.e., for all $e_j \in B^{(1)}$, $j \in \mathbb{I}$, and for all $a^{(2}, b^{(2} \in B^{(2)}$, $c^{(3} \in B^{(3)}$,

$$e_j \neq a^{(2} * b^{(2} \neq c^{(3}.$$

Example 2.1. Let us introduce some standard notations. A limit ordinal $\gamma > 0$ is called indecomposable if there exists no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$. We will take such indecomposable ordinal numbers, which obey equation $\varepsilon = \omega^{\varepsilon}$. The first such ordinal is

$$\varepsilon_0 = 1 + \omega + \omega^{\omega} + \omega^{\omega^{\omega}} + \ldots,$$

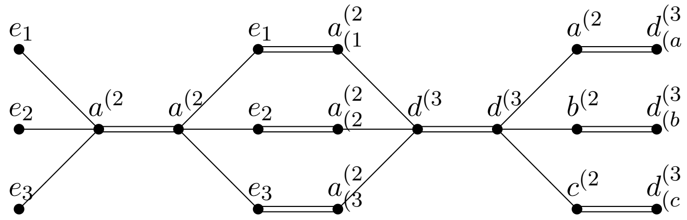
then follow $\varepsilon_1, \varepsilon_2, \ldots$. Now we can write an example for the third level beta group. Let

$$B^{(1)} = \{1, 2, 3, \ldots\},$$

$$B^{(2)} = \{\varepsilon_0, \varepsilon_0 + 1, \varepsilon_0 + 2, \ldots, \varepsilon_0 + \varepsilon_0, \varepsilon_0 + \varepsilon_0 + 1, \ldots\},$$

$$B^{(3)} = \{\varepsilon_1, \varepsilon_1 + 1, \varepsilon_1 + 2, \ldots, \varepsilon_1 + \varepsilon_1, \varepsilon_1 + \varepsilon_1 + 1, \ldots\}.$$

To illustrate visually the concept of the third level beta group we can draw an illustration, where by lines (-) we mean algebraic operation $*$, and by two lines (=) we denote an equality relation. Let $\mathbb{I} = \{1, 2, 3\}$. The illustration must be viewed from the left to the right.



Definition 2.4. *The fourth level beta group* $(B, *)$ *is a set* $B = B^{(1)} \cup B^{(2)} \cup B^{(3)} \cup B^{(4)}$ *together with a binary operation* $*$ *satisfying the following axioms.*

1°-5° *Are the same as in Definition 2.3.*

6° *Each element* $d^{(4} \in B^{(4)}$ *has identity elements* $c^{(3} \in B^{(3)}$*, such that* $c^{(3} * d^{(4} = d^{(4}$ *and inverse elements* $d^{(4}_{(c} \in B^{(4)}$*, such that* $d^{(4} * c^{(3} = d^{(4}_{(c}$.

An example for the fourth level beta group is analogical to the third level beta group, only the set $B^{(4)}$ is added,

$$B^{(4)} = \{\varepsilon_2, \varepsilon_2 + 1, \varepsilon_2 + 2, \ldots, \varepsilon_2 + \varepsilon_0, \ldots, \varepsilon_2 + \varepsilon_1, \ldots, \varepsilon_2 + \varepsilon_2, \ldots\}.$$

Now we are ready to state the general definition of the n-th level beta group.

Definition 2.5. *Let* $n \in \mathbb{N}$*, then the* n*-th level beta group* $(B, *)$ *is a set*

$$B = B^{(1)} \cup B^{(2)} \cup B^{(3)} \cup B^{(4)} \cup ... \cup B^{(n)}$$

together with a binary operation $*$ *satisfying the following axioms.*

1°-6° *Axioms are the same as in Definition 2.4.*

...

n+2° *Each element* $u^{(n} \in B^{(n)}$ *has identity elements* $t^{(n-1} \in B^{(n-1)}$*, such that* $t^{(n-1} * u^{(n} = u^{(n}$ *and inverse elements* $u^{(n}_{(t} \in B^{(n)}$*, such that* $u^{(n} * t^{(n-1} = u^{(n}_{(t}$ *for each* $n \in \mathbb{I}$.

From the last definition straight away follows the next theorem.

Theorem 2.1 *Let* $a^{(n}, b^{(n} \in B^{(n)}$ *and* $n < m$*, then*

$$a^{(n} * b^{(m} = b^{(m} \quad \text{and} \quad a^{(m} * b^{(n} = a^{(m}_{(b}$$

for all $n, m \in \mathbb{N}$.

The last theorem shows us the structure of the leveled beta groups. That is, for n-th level elements all the elements from the $(n-1)$-th levels are identities, and all the elements from the $(n+1)$-th levels are inverse elements (in beta algebra sense). That is, if we are in the n-th level, all elements from the $(n-1)$-th level doesn't make any impact, if they are standing from the left to our element.

Also, let us note that in Definition 2.5 we have chosen $n \in \mathbb{N}$, but the definition could be extended to bigger than $\mathbb{N}$ sets of ordinals. For example, if $n = \omega+5$ then our axiom system will have $\omega+5$ conditions plus closedness and associativity.

Analogical to the previous examples, we can write an example for the n-th level of the leveled beta group, let $n \geq 2$, $n \in \mathbb{N}$,

$$B^{(n)} = \{\varepsilon_{n-2}, \varepsilon_{n-2}+1, \varepsilon_{n-2}+2, \ldots, \varepsilon_{n-2}+\varepsilon_0, \ldots, \varepsilon_{n-2}+\varepsilon_1, \ldots, \varepsilon_{n-2}+\varepsilon_2, \ldots\}.$$

It is easy to see that for the ordinals from our examples we can write

$$B^{(1)} \subset B^{(2)} \subset B^{(3)} \subset \ldots \subset B^{(n)},$$

and, for all $i \in \mathbb{N}$,

$$e_i < a^{(2} < b^{(3} < \ldots < u^{(n},$$

Also it could be easily shown that for all $a^{(j} \in B^{(j)}$, $b^{(j+1}, c^{(j+1} \in B^{(j+1)}$, $d^{(j+2} \in B^{(j+2)}$, and for all $j \in \mathbb{I}$ we have

$$a^{(j} \neq b^{(j+1} * c^{(j+1} \neq d^{(j+2}.$$

The last equation means that we cannot jump from one level to another, if we are performing algebraic operation between elements of the same level.

After we have introduced a concept of the leveled beta groups we can begin to investigate its basic properties. For example let us take a look at equation

$$x * a = b.$$

Let, $k, l, m \in \mathbb{N}$, then the equation in beta algebra has a look

$$x^{(k} * a^{(l} = b^{(m}.$$

The solution $x^{(k}$ of the equation depends on the levels l and m. If $l > m$ then the equation doesn't have any solutions. If $l = m$ then solutions exist if $a^{(l} = b^{(l}$ and k satisfies inequality $k < l$, so $x^{(k} \in B^{(l-1)}$, and we have many solutions. If $l < m$ then solutions exists if $k = m$, and the equation has only one solution for each $a^{(l}$ and $b^{(k}$.

In such a way we can analyse solutions of other equations. We shall leave it to the next papers, where other properties of leveled beta groups will be investigated.

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